MIT OpenCourseWare - Single Variable Calculus (18.01)

Lecture 25: Numerical Integration and Computing π

Introduction

This lecture continues the discussion on numerical methods for integration, focusing on practical examples and applications. The lecture covers:

1. Numerical integration methods (Trapezoidal Rule and Simpson's Rule)
2. A practical example of numerical integration
3. Computing the value of √(π/2)
4. Concluding remarks on the unit

1. Numerical Integration Methods

1.1 Example: Integrating 1/x from 1 to 2

The lecture begins with a practical example of numerical integration, using the integral:

$\int_1^2 \frac{dx}{x}$

This integral has a known analytical solution:

$$\int_1^2 \frac{dx}{x} = \ln(x)\Big|_1^2 = \ln(2) - \ln(1) = \ln(2) \approx 0.693147$$

The professor uses this example to demonstrate numerical integration methods because:

- The exact answer is known (for comparison)
- The function 1/x is relatively simple to evaluate

1.2 Trapezoidal Rule

The Trapezoidal Rule approximates the area under a curve by dividing it into trapezoids.

For this example, the interval [1,2] is divided into just 2 sub-intervals (a very coarse approximation), giving 3 points: x = 1, x = 3/2, and x = 2.

The function values at these points are:

- f(1) = 1
- f(3/2) = 2/3
- f(2) = 1/2

The Trapezoidal Rule formula is:

$$\int_a^b f(x)dx \approx \Delta x \left[\frac{f(x_0)}{2} + f(x_1) + f(x_2) + ... + f(x_{n-1}) + \frac{f(x_n)}{2}\right]$$

Where:

- Δx = (b-a)/n = (2-1)/2 = 1/2
- The pattern of coefficients is (1/2, 1, 1, ..., 1, 1/2)

Applying the formula:

$$\int_1^2 \frac{dx}{x} \approx \frac{1}{2}\left[\frac{1}{2} \cdot 1 + \frac{2}{3} + \frac{1}{2} \cdot \frac{1}{2}\right]$$

The professor mentions that this approximation gives about 0.96, which is quite far from the exact value of 0.693147.

1.3 Simpson's Rule

Simpson's Rule provides a more accurate approximation by fitting parabolas through sets of three points.

The Simpson's Rule formula is:

$$\int_a^b f(x)dx \approx \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

For more points, the pattern of coefficients is (1, 4, 2, 4, 2, ..., 4, 1) for more intervals.

Applying Simpson's Rule to the same example:

$$\int_1^2 \frac{dx}{x} \approx \frac{1}{6}\left[1 + 4 \cdot \frac{2}{3} + \frac{1}{2}\right] = \frac{1}{6}\left[1 + \frac{8}{3} + \frac{1}{2}\right] \approx 0.6944444$$

This gives approximately 0.6944444, which is remarkably close to the exact value of 0.693147.

1.4 Error Analysis for Simpson's Rule

The professor explains that the error in Simpson's Rule is proportional to $(\Delta x)^4$:

$$|Simpson's - Exact| \approx O((\Delta x)^4)$$

This means:

- If Δx = 0.1 (dividing into 10 intervals), the error would be approximately 10^4
- This provides about 4 digits of accuracy with a reasonable number of calculations

Simpson's Rule works so well because:

- It gives exact results for polynomials up to degree 3 (constants, lines, parabolas, and cubics)
- This explains the fourth-order accuracy

1.5 Limitations and Cautions

The professor warns about situations where numerical integration methods may fail:

- When the function has singularities (e.g., 1/x near x = 0)
- When the function or its derivatives are not smooth
- When there's a large amount of area concentrated in a small region

1.6 Remembering the Formulas

A useful way to remember the coefficients in these formulas is to check that they give the exact answer for the simplest case: f(x) = 1.

For the Trapezoidal Rule with f(x) = 1:
$\Delta x\left[\frac{1}{2}+1+1+...+1+\frac{1}{2}\right]=\Delta x\left[\frac{1}{2}+(n-1)+\frac{1}{2}\right]=\Delta x\cdot n=b-a$

This confirms that the Trapezoidal Rule gives the exact area of a rectangle with base (b-a) and height 1.

2. Computing √(π/2)

The second part of the lecture addresses the computation of √(π/2), which the professor describes as "one of the most famous computations in calculus."

2.1 Connection to Previous Work

The professor connects this to previous work on the volume of revolution:

$$V=\int_0^\infty 2\pi re^{-r^2}dr=\pi$$

This was calculated using the shell method, where:

- 2πr is the circumference of the shell
- e^($-r^2$) is the height of the shell

- dr is the thickness of the shell

The calculation proceeds as follows:
$V = \int_0^\infty 2\pi r e^{-r^2} dr = 2\pi \int_0^\infty r e^{-r^2} dr$

Using the substitution $u = r^2$, $du = 2r\,dr$:
$V = 2\pi \int_0^\infty \frac{1}{2} e^{-u} du = \pi \int_0^\infty e^{-u} du = \pi[-e^{-u}]_0^\infty = \pi[0 - (-1)] = \pi$

2.2 The Bell Curve and Its Area

The professor introduces:

$Q = \int_{-\infty}^\infty e^{-x^2} dx$

This represents the area under the bell curve (Gaussian function).

The key insight is that:
$V = Q^2$

Since we already know V = π, this means:
$Q^2 = \pi$
$Q = \sqrt{\pi}$

2.3 Connection to Error Function

The professor connects this to the error function discussed earlier:

$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$

The limit as x approaches infinity:
$\lim_{x\to\infty} \text{erf}(x) = \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = 1$

This confirms that:
$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

Therefore:
$\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$

2.4 Three-Dimensional Visualization

The professor uses a three-dimensional visualization to explain why $V = Q^2$.

The approach involves:

- Setting up a coordinate system with x, y, and z axes

- Visualizing e^(-r²) as a "hump" in three dimensions
- Taking slices along different values of y
- Showing how these slices relate to the original integrals

The key insight is to consider the double integral:
$\iint e^{-(x^2+y^2)} dxdy$

This can be evaluated as:
$\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right) \cdot \left(\int_{-\infty}^{\infty} e^{-y^2} dy\right) = Q \cdot Q = Q^2$

But it can also be evaluated using polar coordinates:
$\int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\theta = 2\pi \int_0^{\infty} r e^{-r^2} dr = \pi$

Therefore, $Q^2 = \pi$, which means $Q = \sqrt{\pi}$.

3. Conclusion

The lecture concludes by tying together the numerical integration methods and the computation of √(π/2), emphasizing the importance of these techniques in calculus and their applications.

The professor mentions that these topics complete the unit on integration techniques, which has covered:

- Basic integration methods
- Integration by parts
- Partial fractions
- Numerical integration
- Applications to volumes and special constants

Key Takeaways

1. **Numerical Integration Methods**:
    - Trapezoidal Rule: Simple but less accurate
    - Simpson's Rule: More accurate, especially for smooth functions
    - Error in Simpson's Rule is proportional to $(\Delta x)^4$
2. **Important Constants**:

- The integral of $e^{(-x^2)}$ from $-\infty$ to ∞ equals $\sqrt{\pi}$
- The integral of $e^{(-x^2)}$ from 0 to ∞ equals $\sqrt{(\pi/2)}$
- These values are fundamental in probability theory and statistics

3. **Integration Techniques**:

- Different methods are appropriate for different types of integrals
- Understanding the limitations of each method is crucial
- Numerical methods provide practical approaches when analytical solutions are difficult

Practice Problems

1. Use the Trapezoidal Rule with n = 4 to approximate $\int_0^1 x^2\,dx$.
2. Use Simpson's Rule with n = 4 to approximate the same integral and compare with the exact answer.
3. Estimate the error in each approximation based on the error formulas discussed.